Cyber-Physical Resilience: Definition and Assessment Metric

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Abstract—Resilient operation of cyber-physical infrastructures in adversarial environments requires \( i \) toughness: maintenance of core crucial sub-functionalities despite ongoing intrusions, and \( ii \) elasticity: recovery of the normal system operation in a timely manner. Put in other words, it does not require unrealistic assumptions about absolute preventative protection of complex cyber-physical platforms that would disable any type of malicious penetration and damage against the physical plant at the first place. Instead, resilience is based on the assumption that a sophisticated intrusion may succeed to evade the deployed protection and runtime detection mechanisms and impact the underlying system services and assets partially (except the core sub-functionalities). The resilient system fights back through reactive and proactive intrusion tolerance mechanisms to respond to ongoing misbehaviors and recover the affected system services and components within a reasonable time interval. In this paper, we present a formal definition of resilience and assessment metric for resilience. Our resilience metric quantifies the ability of the system to recover from an attack provided the attack is discovered within a fixed time interval, as well as the cost of recovery. We analyze the metric properties for linear systems and linear systems with actuator saturation. We then formulate cyber defense policies that ensure the resilience conditions are satisfied and validate our approach using a power system case study.

Index Terms—Cyber-physical systems, cyber security, intrusion resilience.

I. INTRODUCTION

Traditionally, security intrusions would target cyber assets through various attack vectors such as software vulnerability exploitations and social engineering channels. Recent advances in practical deployment of orchestrated operation of the cyber computations and physical processes in safety-critical infrastructures, so-called cyber-physical systems (CPS), have expanded the attack surfaces and their ultimate impact. As shown by the past real incidents, e.g., Stuxnet [1] and BlackEnergy3 [2] malware, cyber-originated misbehaviors can target physical components through a variety of domain-specific attack vectors such as sensor-based data corruptions, malicious control command injection, and the controller compromises.

Recently increasing number and complexity of fast-spreading and high-impact cyber-physical intrusions call for effective protection solutions. Cyber-physical security handling techniques can be categorized into three groups. First, intrusion prevention techniques (e.g., end-to-end data encryption), attempt to ideally prevent any attacks from happening in the first place. Second, intrusion detection solutions (e.g., host-based malware detection by anti-virus solutions) are designed based on the practical assumption that absolute security is infeasible and the attacks will still occur and need to be identified. Finally, intrusion tolerance solutions (e.g., infected asset containment by network firewalls) employ reactive and proactive response and recovery mechanisms to bring the affected system functionalities as the result of the successful and ongoing attacks.

Cyber-physical intrusion resilience aims at \( i \) full correctness maintenance of the core (possibly empty) set of crucial sub-functionalities despite ongoing adversarial misbehaviors. Put in other words, it is acceptable for non-crucial sub-functionalities to be affected (partially degraded or complete failure) temporarily; and \( ii \) guaranteed recovery of the normal operation of the affected sub-functionalities within a predefined cost limit, so-called resilience threshold. The cost limit can be formulated using various criteria such as time (recovery deadline), money (recovery expenses), etc. The abovementioned functionalities may involve integrated sub-components of the cyber network and physical platform.

Consequently, the required protection and Intrusion resilience is an overarching system property that requires a well-integrated and targeted design and deployment of the defense mechanisms (prevention, detection and tolerance) within various components of the target system. To guarantee permanent correct functionality of the crucial services, the corresponding system components have to either implement absolutely secure preventative measures, or leverage tolerance techniques (e.g., redundancy) to eliminate the possibility of any adversarial impact. The remaining components need the security protection mechanisms to an extent that satisfies the resilience threshold.

Our contributions in this paper are as follows:

- We provide a formal and abstract mathematical definition for intrusion resilience in cyber-physical systems. We characterize the resilience metric for linear systems and actuator-saturated linear systems, and provide methodologies for synthesizing controllers that guarantee resilience in such systems.
- We formulate a hierarchical game between the targeted cyber-physical platform and a cyber adversary, in which the value of each state of the game is equal to the resilience of the system. We propose Markovian cyber defense policies for maximizing resilience under cost constraints.
- We evaluate our proposed resilience metric and cyber assessment via power system case studies.

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The rest of the paper is organized as follows. Section II describes the related work. Section III describes the class of attacks considered in this paper. Section IV motivates the need for a definition and metric for cyber-physical intrusion resilience. Section V provides the physical system model, intrusion resilience definition and the corresponding metric. Section VI provides the resilience modeling and metric for the cyber network, and Section VII explains how the cyber and physical side resilience models are used by our framework for a hybrid cyber-physical resilience metric and assessment. Section VIII evaluates our resilience metric in a power system case study. Section II reviews the most related past work in the literature. Finally, Section IX concludes the paper.

II. RELATED WORK

Recent work on adversary-aware control has addressed a range of topics such as models of attack and defense, risk assessment, attack detection and forensics, and secure control design. These contributions move towards development of a principled approach to cyber-physical security of power grid control systems [3].

SOCCA [4] presents a security assessment engine to identify the weaknesses of a smart grid topology against cyber attacks. For improved resiliency of microgrid operations, Che et al. [5] proposes to use a tertiary DC control. SCPSE [6] presents an online smart grid monitoring and incident detection framework to identify malicious misbehaviors on the control network of the power system and perform bad-data detection accordingly. Hassain et al. [7] proposed a resilience analysis technique in a distributed power system control setting to determine the redundant and critical controllers depending on whether the power system maintains its controllability if a particular controller is compromised. Zonouz et al. [8], [9] proposes a embedded system security monitoring solution to ensure the correct operation of the power system controller devices.

Vulnerability assessment of smart grid critical infrastructures has been studied using network interdiction formulations, e.g., sequential games [10], [11], where the operator (leader) chooses a source-destination path, and then the interdictor (follower) inspects one arc to maximize the probability with which the operator is detected. In the model by Bertsimas et al. [12], the operator chooses a feasible flow, and then the interdictor disrupts a fixed number of edges. The interdictor's goal is to minimize the maximum flow that reaches the destination node.

Bienstock [13] develops efficient mixed-integer linear network models for the power system $(N-k)$ problem. Salmeron et al. [14] solved a network interdiction problem to identify maximally disruptive cyber-physical attacks. A resource-constrained attacker (leader) targets a set of control system components, and the defender (follower) uses an optimal-flow model to implement response measures after the attack. The important work by Verma and Bienstock [13] develops network interdiction models to study the $(N-k)$ problem in power grids. State-of-the-art computational methods for solving large-scale mixed integer programs have been applied to solve these problems.

Still a major gap remains to propagate the effect of attacks on smart grids at the level of mathematical control specification to numerical solvers and optimization toolboxes, and runtime machine code executions. Our project aims to build a defense-in-depth resilient design to address this gap. Standard control-theoretic techniques such as robustness aim to ensure that a system is stable in the presence of arbitrary disturbances. The assumptions underlying these approaches are that the disturbances are bounded in energy and that the robust controller is fixed. By comparison, our resilience metric assumes that the disturbance signal can be arbitrary but is of finite duration, and that the controller can be modified to restore stability after a compromise is detected.

Furthermore, the cyber-physical smart-grid operations include close interactions between its cyber assets and physical components. Modeling the involved cyber-physical interdependencies concisely is a challenging endeavor. To that end, we leverage game-theoretic modeling and control theoretic analysis techniques to consider the cyber security interactions between the adversaries and the system, and physical dynamics of the power system, respectively. This hybrid modeling and analysis framework enables us to investigate the effects of any malicious cyber-originated misbehaviors (e.g., controller injection by a computer virus) on the operation of the power system (e.g., whether the generators are dispatched correctly).

III. ATTACK MODEL

Our proposed resilience definition and the corresponding assessment metric mainly focus on attacks that originate from the cyber network and then impact the physical components. The physical plant is assumed to be controlled by a distributed set of (possibly redundant) controllers and actuation points. The (possibly) multi-step cyber attack initiates from an attacker-resident Internet node and traverses the control network through several host compromises via vulnerability exploitations. The adversary’s cyber-side goal is to compromise and gain control over controllers that are in direct contact with the physical plant actuators. Finally, the attacker will leverage the malicious access to the compromised controllers and issue unsafe control commands to drive the underlying physical plant as much away as possible from the safe states.

Once the deployed defense mechanisms detect the compromised controllers, it takes two types of response and recovery actions. As the immediate response, the untrusted controllers are contained and isolated through cyber-side actions (e.g., firewall ruleset updates and remote disconnect commands) so that the compromised controllers cannot affect the underlying physical plant actuators anyways anymore. The defense mechanisms will then take more time-consuming recovery actions (e.g., via restoring a clean state of a compromised controller) to recover normal secure operation of the controllers and connect them back to the network.

IV. MOTIVATION AND OVERVIEW

A. Motivation: Power Grid Case-Study

Traditionally, purely-cyber intrusion resilience solutions could be either model-based [10] that take advantage of system
models for their response strategy optimization, or model-free [15] that do not leverage system models and perform their strategy selection based on sensor data only. In cyber-physical settings, we believe that model-free approaches are often of limited use due to the high system complexity and sophisticated inter-dependencies among the cyber and physical components. For instance, cascading failures are widely studied in the power systems domain and occur due to high interconnectivity among the power assets. As a case in point, a single malicious transmission line outage would indirectly cause several subsequent line outages by forcing power redistribution to adjacent lines, causing them to overload. Using system models, an estimate of the post-outage flows can be calculated using line outage distribution factors [16]. It is noteworthy that similar cascading overload scenarios could be caused by an ill-designed intrusion resilience engine that attempts to fix a problem locally without consideration of the action’s global impact on the power network. Model-based techniques for cyber-physical intrusion resilience can potentially consider such complex failure and recovery scenarios.

1) Integrated cyber-physical resilience: Cyber-based resilience represents strategies and actions that deal with cyber components and their recovery from intrusions. For instance, a firewall reconfiguration to proactively prevent an upcoming attack is a cyber-based response action. Power-side intrusion response actions support corrective manipulation of power components, e.g., generation redispatch or line status changes to tolerate a recent malicious line outage. Power-based strategies and actions support corrective manipulation of power components for resilience purposes such as power topology changes to tolerate a recent malicious line outage. When responding to and recovering from intrusions, cyber- and power-based resilience engines deal with two completely different types of system dynamics and incidents (discrete sequential logic within a computing platform vs. continuous differential dynamics governing the physical power components). The actions taken by cyber- and power-based resilience engines are also radically different. The difference arises because physical actions often occur on a continuum, e.g., a real number representing the power generation set-point, whereas cyber systems have a discrete action set, e.g., block/allow access attempts to a particular system file.

A truly comprehensive cyber-physical intrusion resilience architecture makes use of both cyber and physical resilience engines in an integrated manner such that both discrete and continuous dynamics are taken into account. Equivalently, it requires extensions to i) cyber-based intrusion resilience to make them power-aware so that they take into account the power system dynamics and topologies when deciding upon a cyber-based action. For instance, a cyber-based resilience engine within a CPS setting may prioritize recovery of a crashed cyber host in charge of a critical generator control over an unavailable historian logging server host because its failure leads to more severe physical impact; ii) power-based fault resilience to make them cyber-security-aware such that they make operating decisions on optimal response strategies considering the cyber network status. For instance, a power-based resilience engine within a CPS setting may choose to isolate a particular generator from the rest of the power system (e.g., through node-breaker reconfiguration at the substation) and compensate for its missing power through secondary generation plants after receiving a recent notification that the first generator’s controller has been compromised.

Why consider physical system resilience actions as opposed to pure reliance on cyber-side capabilities? Comprehensive cyber-physical intrusion resilience requires powerside response actions because of the following reasons: i) A malicious attacker may break down a power component such that the system cannot be restored without physical power-side actions, e.g., by causing a generator to blow up. When cyber-only capabilities cannot fix the compromise, automated fixing via cyber-only capabilities is not possible, taking physical power-side actions is necessary, e.g., switching to a redundant component. ii) The consequence of a malicious attack on the power side is occasionally fixable through automated commands, e.g., a malicious relay opening could be reverted simply through a cyber-side close command. However, the attacker may open the relay again if the controller remains compromised and its clean state cannot be fully recovered. In those cases, cyber-side restoration is not feasible based on the cyber system’s built-in capabilities and degree of redundancy, and power-side response actions may be required. If the cyber side restoration is not feasible based on the system’s built-in capabilities and degree of redundancy. For instance, power-side actions may be taken to physically isolate the compromised controller through power topology reconfiguration using other non-compromised breakers in the substation.

Why consider cyber-side response actions as opposed to pure reliance on traditional control? This is the dual problem to the one discussed above. Cyber-physical intrusion resilience requires cyber-based response and recovery action execution because of the following reasons: i) Pure reliance on power-side fault resilience may be too costly and slow for practical deployments. Power-aware cyber-side resilience facilitates execution of corrective actions on the cyber side to restore the physical system after an attack (e.g., recover from a malicious relay opening) through timely restoration of the compromised controller. In our example attack above, once the cyber component has been recovered, the protective relay can simply re-close. Without such cyber-side resilience support, any potential malicious relay manipulation on a line would require physical power system reconfiguration and/or redispatch. ii) Permanent recovery from a cyber-originated cyber-physical intrusion requires cyber resilience mechanisms; otherwise, each time the physical resilience solution fixes an attack consequence, e.g., re-closure of an opened circuit breaker, the attacker could immediately cause the same consequence again, because the physical resilience engine is not aware of the compromised set of cyber assets and can only take cyber-blind control actions that are never able to “clean” the system from malicious parties or patch the cyber vulnerabilities. Consequently, without knowledge of the cyber state, any compromises and their impacts are essentially cumulative, due to the fact that the system is never cleaned. The system would only become more compromised, never less, and the physical-only resilience engine would have an increasingly
and assessment algorithm.

B. Resilience Metric Overview

Our approach to CPS resilience is illustrated in Figure 1. The system is a hierarchy with two layers. The top (cyber) layer represents the progress of the cyber attack and is represented as a Markov decision process. The bottom (physical) layer represents the impact of the attack on the physical plant, and plan to integrate observer-based detection of attacks into our approach in future work.

B. Metric Definition

In order to define the metric, we first introduce the notion of basin of attraction.

Definition 1: The basin of attraction is the set of states $D$ such that $x(0) \in D$ and $u(t) = v(t) = 0$ implies that $\lim_{t\to\infty} x(t) = 0$. Intuitively, the basin of attraction is defined as the set of initial states such that the system will return to the equilibrium point 0 if no additional input is provided. In addition to the basin $D$, we define the a set of safe states $S$. Safe states represent constraints on the basic functionality of the system, e.g., ensuring that rotor angle separations in a power system do not exceed $\pi/2$. We let $W = S \cap D$. The definition of resilience is given as follows.

Definition 2: Let $T$ denote the time when a system compromise is detected and removed. A system $(\Omega)$ is resilient if, for any adversarial input $\{v(t) : t \in [0, T]\}$, the resulting state $x(T)$ lies in the basin of attraction when $x(0) = 0$.

This definition of resilience is analogous to the definition of resilience in materials science, namely, the amount of energy that must be exerted to steer the system to a state from which it cannot recover to the stable equilibrium. We further define the cost of recovery as follows.

Definition 3: Let $\mathcal{R}$ denote the set of states $x(T)$ that are reachable by the adversary within time $T$. For any state $x_0 \in \mathcal{R}$, define $\mathcal{E}(x_0)$ to be the cost required to steer the system state to the origin, i.e.,

$$\mathcal{E}(x_0) \triangleq \min \left\{ \int_0^T x(t)^T Q x(t) + u(t)^T Ru(t) : \lim_{t\to\infty} x(t) = 0, x(0) = x_0 \right\}$$

where $Q$ and $R$ are positive definite matrices that describe the cost of deviating from the zero state and the cost of control, respectively. The cost of recovery is defined by

$$\max \{ \mathcal{E}(x) : x \in \mathcal{R} \}.$$ 

C. Resilience of Linear Systems

We first consider linear systems, which will provide initial insights towards our approach. The results of this subsection are valid for linear systems, but may also be applicable to nonlinear systems that can be approximated by linear systems, such as the linearization of a nonlinear system around a stable operating point. This approach to analysis of nonlinear systems
is relevant to scenarios such as small-signal stability evaluation in power systems. For a linear system with dynamics

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
\dot{y}(t) &= x(t)
\end{align*}
\]

the conditions for resilience can be described via controllability analysis of the system. Consider the controller compromised attack described by (2). As a preliminary, let \(\{q_1, \ldots, q_l\}\) denote the set of unstable modes of \(A\).

**Proposition 1:** Suppose that \((A, B)\) is a linear system and that \(A\) is diagonalizable. The system (3) is resilient to compromise of a set of controllers \(S\) iff any unstable mode \(q\) of \(A\) that lies in the controllable subspace of \(E(S)\) also lies in the controllable subspace of \(B(S)\).

**Proof:** Suppose that an unstable mode \(q\) lies in the controllable subspace of \(E(S)\) but not the controllable subspace of \(B(S)\). Then there is an input \(\{v(t) : t \in [0, T]\}\) such that \(x(T) = q\) when \(u(t) = 0\). Furthermore, by superposition and the fact that \(q\) is not in the controllable subspace of \(B(S)\), for any input \(u(t)\), \(x(T) = q + r\) where \(r\) is linearly independent of \(q\). Since \(q\) is not in the controllable subspace of \((A, B(S))\), \(q\) is not in the span of the controllability Gramian for the system \((A, B(S))\), the vector \(x(T)\) is also not in the span, and hence there is no input \(u(t)\) that can steer the system to a neighborhood of the origin.

Conversely, suppose that every unstable mode that is in the controllable subspace of \((A, E(S))\) is also in the controllable subspace of \((A, B(S))\). Then \(x(T)\) can be decomposed as \(x(T) = r' + r''\), where \(r'\) is in the controllable subspace and \(r''\) is in the span of the stable modes, and hence will converge to 0 in the absence of any control action. Thus for any \(\epsilon > 0\), there exists a time index \(T'\) and a control signal \(\{u(t) : t \in [T, T']\}\) such that \(|x(T')| < \epsilon\), implying that convergence to 0 can be guaranteed.

Based on Proposition 1, we say that a set of controllers \(Z\) is resilient if the set of unstable modes, denoted \(\{u_1, \ldots, u_l\}\), satisfy \(\{u_1, \ldots, u_l\} \subset \text{span}(B(Z) \ A B(Z) \ \cdots \ A^{n-1} B(Z))\). A set of controllers is denoted a critical resilient set if all non-trivial subsets of \(Z\) are not resilient. Equivalently, \(Z\) is a critical resilient set if for any \(j \in Z\), there exists \(i \in \{1, \ldots, l\}\) such that \(u_i \notin \text{span}(B(Z \ \{ j \}) \ \cdots \ A^{n-1} B(Z \ \{ j \}))\). The problem of selecting a minimum-cardinality set of controllers satisfying this condition has been considered in, e.g., [17], and is known to be NP-hard but approximable within polynomial time up to a provable optimality bound of \(O(\log m)\).

The cost of recovery for a linear system can be computed by minimizing the cost function

\[
\max_{x(0)} \left\{ \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) \ dt \right\}
\]

The controller that minimizes this cost function is a feedback controller \(u(t) = K(S)x(t)\), where \(K(S)\) is obtained as \(K(S) = -R^{-1}Q + P\) and \(P\) is the solution to the algebraic Riccati equation

\[
A^T P + PA - PB(S)R^{-1}B(S)^T P + Q = 0.
\]

**D. Resilience under Actuator Saturation**

Another relevant case is linear systems with actuator saturation, which have dynamics defined by

\[
\dot{x}(t) = Ax(t) + B \text{sat}(u(t)),
\]

where \(\text{sat}: \mathbb{R}^m \rightarrow \mathbb{R}^m\) is a function defined by

\[
\text{sat}(u)(i) = \begin{cases} 
-\gamma, & u_i < -\gamma \\
\gamma, & u_i > \gamma \\
u_i, & \text{else}
\end{cases}
\]
Actuator saturation is relevant to resilience because it provides a natural scenario in which the adversary has a limited impact on the system, while the system has a limit on its ability to recover. For ease of analysis, we assume that only a single controller is compromised; our approach, however, can be extended to compromise of multiple controllers.

The resilient system design must ensure that, if the adversary is detected within time $T$, the remaining non-compromised controllers can still drive the system state back to 0. Our approach is to synthesize a family of resilient controllers $u(t) = Kx(t)$ such that the system state is restricted to a region where the actuator saturation constraints are never binding, or equivalently, $||u(t)||_\infty \leq \gamma$. We restrict to this set of states in order to ensure computational tractability of the controller design problem. Specifically, we synthesize a single controller matrix $K$ and a region $\mathcal{R}$ that guarantees that the system remains in region $\mathcal{R}$ up to time $T$ after a single controller has been compromised, as well as a set of controllers $K_1, \ldots, K_m$ that steer the state back to the origin after the compromise has been detected. Under this approach, the state dynamics are given by

$$x(t) = \begin{cases} Ax(t) + B_{i\text{sat}}(Kx(t)), & t \in [0, T] \\ (A + B_{i\text{sat}}(K_i)x(t)), & t > T \end{cases}$$

where $B_i$ is equal to the $i$-th column of $B$ with the $i$-th element (corresponding to the compromised controller) set to 0.

**Proposition 2:** Suppose that there exists a set $\mathcal{R}$ and a set of controllers $K_1, \ldots, K_m$ and $K$ satisfying the following conditions: (a) If node $i$ is compromised at time 0 with $x(0) = 0$, then $\{x(t) : t \in [0, T]\} \subset \mathcal{R}$ when controller $K$ is active; (b) Controller $K_i$ guarantees asymptotic stability of the origin and positive invariance of $\mathcal{R}$; (c) $||K_i x||_\infty \leq \gamma$ and $||K_i x||_\infty \leq \gamma$ for all $x \in \mathcal{R}$, where $K_i$ is the matrix obtained by removing the $i$-th column of $K$. Then the system is resilient to compromise of any single controller.

**Proof:** If conditions (a)–(c) hold, then the system dynamics are given by

$$\dot{x}(t) = \begin{cases} (A + B_i K_i)x(t) + E\text{sat}(v), & t \in [0, T] \\ (A + B_i K_i)x(t), & t > T \end{cases}$$

where $E\text{sat}(v)$ represents the impact of the malicious controller. Hence for $t > T$, the fact that $(A + B_i K_i)$ is asymptotically stable guarantees that the system state returns to 0 and resilience is satisfied.

In order to satisfy the conditions of Proposition 2, we consider ellipsoidal $\mathcal{R}$ defined by

$$\mathcal{R} := \{ x : x^T P x \leq \frac{1}{\alpha^2} \},$$

where $\alpha > 0$ and $P$ is a positive definite matrix. We compute a candidate value of $P$ by solving

$$A^T P + P A - \rho P = -I,$$

where $\rho$ is the largest eigenvalue of $A$.

We choose the matrix $P$ based on (4) so that we can apply Lyapunov theory to prove invariance of the set $\mathcal{R}$, while the goal of minimizing $\rho$ can be interpreted as selecting the minimum-energy controller needed to stabilize $A$.

**Remark 1:** Certain types of safety constraint can also be incorporated into this choice of $\mathcal{R}$. For example, when the set of safe states is an ellipsoidal region $\{x^T \Theta x \leq \xi\}$, the set $\{x^T P x \leq \frac{1}{\alpha^2}\}$ satisfies the safety requirement iff

$$\alpha \geq \frac{1}{\sqrt{\xi}} \lambda_{\text{max}}((P^{-1/2})^T \Theta P^{-1/2}).$$

Using this definition of $P$, it remains to find $\mathcal{R}$ and the controller matrices. Turning to property (a) of Proposition 2, we have the following sufficient condition.

**Lemma 1:** Suppose that $\mathcal{R} = \{ x : x^T P x \leq \frac{1}{\alpha^2}\}$ and there exists $b > 0$ such that for any $x \in \mathcal{R}$,

$$x^T (A_i^T P + P A_i)x + 2b E_i^T P x \leq \frac{1}{\alpha^2 T}.$$  

Then $\{x(t) : t \in [0, T]\} \subset \mathcal{R}$ for any adversarial input signal $v(t)$.

**Proof:** Define a function $V(t)$ by $V(t) = x(t)^T P x(t)$. Since $V(0) = 0$, a sufficient condition for $\{x(t) : t \in [0, T]\} \subset \mathcal{R}$ is $V(t) \leq \frac{1}{\alpha^2 T}$ for $t \in [0, T]$. We have that

$$V(t) = x(t)^T P x(t) + \dot{x}(t)^T P x(t) = x(t)^T (A_i^T P + P A_i)x(t) + 2v(t) E_i^T P x$$

yielding the desired result.

Lemma 1 implies that

$$\max \{x^T (A_i^T P + P A_i)x + 2v E_i^T P x : x^T P x \leq \frac{1}{\alpha^2 T}, |v| \leq b\} \leq \frac{1}{\alpha^2 T}$$

is a sufficient condition for (a) in Proposition 2. Since the maximum value occurs at $v = b$ if $E_i^T P x > 0$ and $v = -b$ otherwise, we obtain that this condition is equivalent to

$$\max \{x^T (A_i^T P + P A_i)x + 2b E_i^T P x : x^T P x \leq \frac{1}{\alpha^2}\} \leq \frac{1}{\alpha^2 T}.$$  

Finally, letting $\dot{x} = \alpha x$ and multiplying both sides by $\alpha^2$ gives the equivalent condition

$$\max \{x^T (A_i^T P + P A_i)x + 2\alpha b E_i^T P x : x^T P x \leq 1\} \leq \frac{1}{T}$$  

The left-hand side of (5) is a pointwise maximum of convex functions in $A_i$ and $\alpha$, and hence is convex, making this a convex constraint for (a).

For condition (b) of Proposition 2, a sufficient condition is that the matrix $A_i := A + B_i K_i$ satisfies the Lyapunov equation

$$A_i^T P + P A_i < 0,$$

which is a linear matrix constraint. Finally, the condition

$$\max \{||K x||_\infty : x^T P x \leq 1\}$$

can be made equivalent to

$$\max \{||K x||_\infty : \dot{x}^T P \dot{x} \leq 1\} \leq \gamma \alpha,$$

which is convex in $K_i$ and $\alpha$, by setting $\dot{x} = \alpha x$. Combining
these conditions together yields the convex feasibility problem

variables: \( \alpha, K, K_1, \ldots, K_m, \bar{A}_1, \ldots, \bar{A}_m, A_1, \ldots, A_m \)

constraints: \( A_i = A + B_i K, \bar{A}_i = A + B_i \bar{K}, i = 1, \ldots, m \)
\[
\max \{\|\dot{x}(t)\|^2 (\dot{Q} + \bar{K}_i^T R \bar{K}_i) \| x(t) \| \leq \sigma \}
\]
\[
\bar{A}_i^T P + PA_i < 0, \bar{A}_i^T P + PA_i < 0, i = 1, \ldots, m
\]
\[
\max \{\|K_i\|_\infty : \|x\|^2 P x \leq 1\} \leq \alpha \gamma
\]
\[
\max \{\|K_i\|_\infty : \|x\|^2 P x \leq 1\} \leq \alpha \gamma
\]

Minimizing the cost of recovery is described as follows. Suppose the cost function is defined by
\[
U = \max_{x(0) \in \mathbb{R}} \int_0^\infty x(t)^T Q x(t) + u(t) R u(t) \, dt.
\]

A bound on the cost function is given by the following lemma.

**Lemma 2:** Suppose that
\[
\max_{i=1,\ldots,m} \sigma_{\max}(P^{-1/2} (Q + \bar{K}_i^T R \bar{K}_i) P^{-1/2}) \leq \sigma
\]
and \( \dot{V}(t) \leq -\rho V(t) \), where \( V(t) = x(t)^T P x(t) \). Then \( U \leq \frac{1}{2 \sigma \rho} \).

**Proof:** For any \( i = 1, \ldots, m \), we have that
\[
U = \max_{x(0) \in \mathbb{R}} \int_0^\infty x(t)^T Q x(t) + u(t) R u(t) \, dt
\]
\[
= \max_{x(0) \in \mathbb{R}} \int_0^\infty x(t)^T P^{-1/2} (Q + \bar{K}_i^T R \bar{K}_i) P^{-1/2} P^{-1/2} x(t) \, dt
\]
\[
\leq \max_{x(0) \in \mathbb{R}} \int_0^\infty x(t)^T P x(t) \sigma_{\max}(P^{-1/2} (Q + \bar{K}_i^T R \bar{K}_i) P^{-1/2}) \, dt
\]
which follows from the definition of the singular value. Furthermore, if \( \dot{V}(t) \leq -\rho V(t) \), then \( x(t)^T P x(t) = V(t) \leq e^{-\rho t} x(0)^T P x(0) \). Combining these inequalities yields
\[
U \leq \max_{x(0) \in \mathbb{R}} \int_0^\infty e^{-\rho t} x(0)^T P x(0) \sigma_{\max}(P^{-1/2} (Q + \bar{K}_i^T R \bar{K}_i) P^{-1/2}) \, dt
\]
By definition \( \mathcal{R} = \{ x : \|x\|^2 P x \leq \frac{1}{c} \} \), and thus \( U \leq \frac{1}{c \sigma \rho} \).

Lemma 2 implies that, in order to minimize the cost of recovery, it suffices to find the minimum value of \( \beta := \frac{\sigma}{\rho} \) such that
\[
\sigma_{\max}(P^{-1/2} (Q + \bar{K}_i^T R \bar{K}_i) P^{-1/2}) \leq \sigma
\]
\[
\dot{V}(t) \leq -\rho V(t)
\]
We have that (8) is equivalent to
\[
P^{-1/2} (Q + \bar{K}_i^T R \bar{K}_i) P^{-1/2} - \alpha I \leq 0,
\]
which is in turn equivalent to \( \sigma P - Q - \bar{K}_i^T R \bar{K}_i \leq 0 \). Using the Schur complement theorem, this inequality is equivalent to the linear matrix inequality
\[
\left( \begin{array}{cc} \sigma P - Q & \bar{K}_i^T \\ R & -1 \end{array} \right) \geq 0.
\]
For constraint (9), we have that \( \dot{V}(t) \leq -\rho V(t) \) is equivalent to
\[
(A + B_i \bar{K}_i)^T P + P(A + B_i \bar{K}_i) + \rho P \leq 0,
\]
which is a linear matrix inequality in \( \rho \) and \( \bar{K}_i \).

Combining these inequalities with (6) yields a formulation for synthesizing resilient controllers with minimum cost of recovery.

Eq. (10) is convex for any fixed \( \beta \); hence, an optimal solution can be found in quasi-polynomial time by using a bisection algorithm to iterate over the possible values of \( \beta \).

As an example, consider a single-input, single-output system with one state variable. The dynamics are given by
\[
\dot{x}(t) = ax(t) + bsat(u_1(t)) + bsat(u_2(t)),
\]
where \( a \) and \( b \) are positive constants. The system is resilient if the system can always be recovered to the 0 state when the attack is detected within time \( T \). Suppose without loss of generality that the signal \( u_2(t) \) is compromised first. We observe that if \( x(0) > 0 \), then the adversary’s optimal strategy will be to choose \( u_2(t) = \gamma \), while the best-case for the system response will be \( u_1(t) = -\gamma \). The resulting dynamics are given by \( \dot{x}(t) = ax(t) \), so that \( x(t) = e^{at} x(0) \).

Now, we consider the set of states \( x \) such that the system can be returned to 0 from state \( x \) after the attack is detected. We have that \( \dot{x}(t) > ax(t) - b \), and therefore the system can be returned to 0 if \( ax < b \), or alternatively \( x < \frac{b}{a} \). Hence, if \( x(T) > \frac{b}{a} \) at the detection time \( T \), then the system state cannot be returned to 0. Equivalently, if \( x(0) > e^{-at} \frac{b}{a} \), then the adversary can destabilize the system.

We therefore arrive at the following control strategy to ensure resilience. When \( |x(t)| < e^{-at} \frac{b}{a} - \varepsilon \) for some \( \varepsilon > 0 \), then the system follows a stabilizing linear control law, i.e.,
\[
u_1(t) = u_2(t) = -\frac{(a+\delta)}{b} x(t)
\]
for some \( \delta > 0 \). For \( x(t) \) outside of that range, follow the law \( u_1(t) = u_2(t) = \gamma \) when \( x(t) < 0 \) and \( u_1(t) = u_2(t) = -\gamma \) when \( x(t) > 0 \). Following such a rule will ensure resilience provided that the system is initialized close to 0.

**VI. CYBER-SIDE RESILIENCE**

We explain how we model cyber network security attacks using a hierarchical game scheme for resilience assessment.

**A. Defense vs. Attacker Modeling**

Generally, every cyber attack path consists of an escalating series of vulnerability exploitations by the adversary. The adversary initially has no access to the control network and then achieves the privilege required to reach his or her malicious goals, e.g., causing a power transmission line outage by opening the corresponding relay. In particular, a state is defined
as the compromised privilege domains in that state. Therefore, the initial state is $(s)$, in which the attacker does not yet have any privileges over the power network. Each adversarial state transition represents a privilege escalation which is achieved through a vulnerability exploitation.

Reciprocal interaction between the adversary and system operator or automated response engine is a game in which each player tries to maximize his or her own benefit. Formally, the response selection process by system operators is modeled as a sequential Stackelberg stochastic game [18] in which the operator acts as the leader while the attacker is the follower; however, in our infinite-horizon game model, their roles may change without affecting the final solution to the problem.

More specifically, the game is a finite set of security states $\mathcal{A}$ that cover all possible security conditions that the system could be in. The system is in one of the security states $s$ at each time instant. From the system’s current state, the operator, i.e., the leader, chooses and takes a response action $a_t \in \mathcal{A}$ admissible in $s$, which leads to a security state transition to $s’$. The attacker, which is the follower, observes the action selected by the leader, and then chooses and takes an adversarial action $a_b \in \mathcal{A}$ admissible in $s’$, resulting in a state transition to $s''$. At each transition stage, players may receive some reward according to the reward function for each player at the destination state. The reward function value for the operator is defined as the security measure of the corresponding state. On the other hand, the reward function for an attacker is usually not known accurately, because an attacker’s reward depends on his final malicious goal, which is also not always known. Therefore, assuming that the attacker takes the worst possible adversarial action, the response actions are chosen based on a security strategy called maximin (discussed later). It is also important to note here that although $S$ is a finite set, it is possible for the game to revert back to some previous state; therefore, the operator-adversary game can theoretically continue forever. This stochastic game is essentially an antagonistic multi-controller Markov decision process, called a competitive Markov decision process (CMDP) [19].

Formally, a discrete competitive Markovian decision process $\Gamma$ is defined as a tuple $(S, A, Res(\cdot), P, \gamma)$ where $S$ is the security state space, assumed to be an arbitrary non-empty set endowed with the discrete topology. $A$ is the set of actions, which itself is partitioned into response actions and adversarial actions depending on the player. For every $s \in S$, $A(s) \subset A$ is the set of admissible actions at state $s$. The measurable function $Sec: S \rightarrow [0, 1]$ is the security measure calculated for each state, and $P$ is the transition probability function; that is, if the present state of the system is $s \in S$ and an action $a \in A(s)$ is taken, resulting in state transition to state $s’$ with probability $P(s'|s, a)$, an immediate reward $Res(s’)$, i.e., resilience measure value of the state $s’$, is obtained by the player taking the action. $\gamma$ is the discount factor and is normalized, i.e., $0 < \gamma < 1$.

## B. Optimal response selection

We explain how we model the response action selection procedure by the defense mechanism. Our solution solves the generated CMDP to find the optimal action which maximizes the expected accumulative long-run reward measure received after a sequence of response and adversarial actions. Using the infinite-horizon discounted cost technique [20], the solution gives more weight to nearer future rewards by recursively adding up the immediate reward, i.e., resilience measure value $Res(\cdot)$, and the discounted expected game value from then on.

To formulate, we compute the optimal policy $\pi^*$ that associates with any belief state $b \in B$ an optimal action $\pi^*(b)$. Our solution formulates the response action selection procedure as a game-theoretic maximin problem. In particular, every policy $\pi$ is assigned a value function $V_\pi$ that associates every belief state $b \in B$ with an expected global reward $V_\pi(b)$ obtained by applying $\pi$ in $b$. For finite-horizon POMDPs, the optimal value function is piecewise-linear and convex [21], and it can be represented as a finite set of vectors. In the infinite-horizon formulation, a finite vector set can closely approximate the optimal value function $V^*$, whose shape remains convex. Bellman’s optimality equation (Equation (11)) characterizes the unique optimal value function $V^*$, from which an optimal policy $\pi^*$ can be easily derived:

$$ V^*(b) = \max_{a_t \in A(b)} \Psi(V^*, b, a_t), $$  \hspace{1cm} (11)

where $\Psi$ denotes the value function given that a specific response action is taken:

$$ \Psi(V^*, b, a) = \min_{a_b \in A(b)} [p(b_{t, a}^n) + \sqrt{\gamma} \cdot \min_{a_t \in A(b)} \{p(b_{t, a}^n) + \sqrt{\gamma} \cdot V(b_{t, a}^n)\}], $$  \hspace{1cm} (12)

in which $b_{t, a}^n$ denotes the updated next belief state if the current state is $b$ and action $a$ is taken:

$$ b_{t, a}^n(s') = \sum_{s \in S} [P(s'|s, a) \cdot b(s)], $$  \hspace{1cm} (13)

and the $p$ function computes security measure values for belief states using security levels of individual states:

$$ p(b) = \sum_{s \in S} [b(s) \cdot Res(s)]. $$  \hspace{1cm} (14)

Briefly, to calculate $V^*$ numerically, our solution uses the value iteration algorithm [22] that applies dynamic programming iterative updates to gradually improve on the value until it converges to the $\epsilon$-optimal value function [22], i.e. $| V_\epsilon(b) - V_{\epsilon-1}(b) | < \epsilon$. Through improvement of the value, the policy is implicitly improved as well. Once the partially observable decision process is formulated and the $\epsilon$-optimal value function is calculated, the solution determines the optimal response strategy $\pi^*$ at any given belief state by choosing the response action which maximizes $V^*$. The optimal policy $\pi^*$ maps the system’s current belief state $b$ to a response action using the following equation:

$$ \pi^*(b) = \arg \max_{a_t \in A(b)} \Psi(V^*, b, a_t). $$  \hspace{1cm} (15)

## C. Automatic Cyber Model Generation

We discuss how one can generate the game-theoretic CMDP for the target control network given the network topology, access control policies and cyber-physical interconnections among cyber and physical components.
The control network’s access control policies, such as firewall rulesets, are composed of rules about sources (IP/port addresses) that are either allowed or not allowed to reach a destination. Our implementation parses the rulesets and creates a binary network connectivity matrix that is a Cartesian product of host systems. The \([i,j]\) entry of the matrix takes on a true value if traffic from host \(h_i\) to host \(h_j\) is allowed, and a false value if it is not allowed. The connectivity matrix always includes an Internet node representing a group of hosts outside of the network where attackers are assumed to initially reside.

The implemented solution generates a comprehensive CMDP model of the control network that represents all possible attack paths. In particular, the generated CMDP by design, would address all system vulnerabilities including previously unknown exploitations. Additionally, the monotonicity property \([23]\) is assumed; in other words, an attacker never backtracks, and hence does not need to relinquish privileges already gained.

To generate the CMDP model, our tool analyses the control network topology input to find out about the set of known system vulnerabilities and individual host computers, i.e., privilege domains. Given the set of system vulnerabilities, the connectivity matrix is updated accordingly to encode adversarial paths only. In particular, the tool automatically generates a CMDP by traversing the connectivity matrix and concurrently updating the CMDP. First, our solution creates the CMDP’s initial state \((\emptyset)\) and starts the CMDP generation with the network’s entry point (Internet) node in the connectivity matrix. Considering the connectivity matrix as a directed graph, the solution runs a depth-first search (DFS) on the graph. While DFS is recursively traversing the graph, it keeps track of the current state in the CMDP, i.e., the set of privileges already gained through the path traversed so far by DFS. When DFS meets a graph edge \([i,j]\) that crosses over privilege domains \(h_i\) to \(h_j\), a state transition \(a_{ij} \in A\) in CMDP is created if the current state in CMDP does not include the privilege domain of the host to which the edge leads, i.e., \(h_j\). The transition in CMDP is between the current state and the state that includes exactly the same privilege set as the current state plus the host \(h_j\), directed by the graph edge \([i,j]\). The CMDP’s current state in the algorithm is then updated to the latter state, and the algorithm proceeds until no further updates to CMDP are possible according to the connectivity matrix.

In addition to the adversarial transitions, the above algorithm also updates the CMDP regarding possible response and recovery actions \(a_e \in A\). In particular, host redundancies, specified by the control network topology input, help to create responsive state transitions. As a case point, consider that for a control network data historian server in the control network there exists a redundant hot spare server designated for intrusion tolerance purposes. To model such a proactive design, our solution creates a responsive state transition, denoting the recover the historian server action, from any state in which the historian server is compromised to states containing the same privileges except the historian server. At that point, the offline CMDP generation is complete, and by design, the CMDP includes all possible attack paths launching from remote (Internet) host systems against the network as well as response and recovery scenarios.

VII. HYBRID CPS RESILIENCE METRIC

Our cyber-physical resilience metric employs the individual resilience metrics for the physical plant (Section V) and the cyber network (Section VI) to evaluate the resilience of the whole cyber-physical platform against the malicious cyber-originated misbehaviors.

Section V presented a resilience metric that considers only the physical components given the subset of compromised controllers regardless of how those controllers get compromised. The difficulty of compromising a specific subset of controllers heavily depends on the cyber network topology and its configuration and global access control policies. Section VI-A modeled this through the state-based CMDP model generation and analysis. The model can also be enhanced through cyber security intrusion detection sensor alerts (observable incidents), and treated as a hidden Markovian model (HMM) to estimate the current state of the under-attack cyber network given the last sequence of the triggered alerts\(^1\).

On the other hand, although the cyber-side network modeling and analysis can assess the difficulty of compromising any subset of controllers through cyber attack paths, it lacks the information regarding the impact of those compromised controllers on the resilience of the physical plant, which is measured by the resilience metric discussed in Section V. Section VI assumed the impact measure given as the function \(Res : S \rightarrow \mathcal{R}\).

In particular, the impact of the compromised controllers on the physical plant is measured by the recovery cost \(U\) formulated in Equation 7. Therefore, through the following assignment, our framework’s cyber-physical resilience metric considers both cyber and physical topologies:

\[
Res(s) \leftarrow -U_s, \tag{16}
\]

where \(U_s\) denotes the recovery cost of the physical system for the case, where the subset of the controllers defined by \(s \in S\) (in CMDP) have been compromised. Using the abovementioned assignment, the cyber-based response system can leverage the measured impact of the compromised controllers defined by the CMDP’s state notion (defined in Section VI-C; shown on Figure 1).

As a result, our framework can measure the resilience metric \(Res\) for individual states of the generated CMDP model \(s_i \in S\) by computing the physical system metric \(U_{s_i}\) for each state. It is possible that, in the CMDP state \(s_i\), none of the controllers may be compromised yet, because the adversary has not penetrated into the cyber control network sufficiently deeply. For instance, in Figure 1, the CMDP state \(s_3 = \{D,E\}\) does not include a compromised controller, whereas, one of its subsequent states \(s_5 = \{D,E,A\}\) includes a compromised

\(^1\)Such a cyber security state estimation can be accomplished using conventional HMM techniques (e.g., Viterbi-based most likely path determination) and is outside of the scope of our paper.
controller A. The state transition \( s_3 \to s_5 \) requires compromising a software vulnerability in controller A by the adversary, whereas the state transition \( s_5 \to s_3 \) requires recovery of the compromised controller A through restoring its clean state by the response system.

For the CMDP states \( s \) with no compromised controllers, our solution assigns \( \text{Res}(s) \leftarrow 0 \). For the states with a nonempty subset of compromised controllers, we use Equation 16 to calculate the resilience measure for the state. Once, the \( \text{Res} \) function are evaluated for all the states, our solution will implement the value iteration procedures to calculate the optimal value function \( V : S \to \mathbb{R} \) using Equation 11. Once the value iteration converges, the calculated values for individual CMDP model states represent the cyber-physical resilience metric for individual states:

\[
\text{CPR}(s) = V^*(s) \quad \forall s \in S.
\]  

(17)

The evaluated cyber-physical resilience metric CPR\((s)\) involves \( i \) the difficulty of the cyber attack paths starting from the current state \( s \) to gain control over any subset of the controllers with actuation power on the physical plant, and \( ii \) the ultimate impact of the those attacks (if the path is successfully traversed) on the physical plant’s resilience (i.e., whether the physical system’s safe state can be recovered, and if so, what the recovery cost would be).

VIII. Evaluation

In this section, we discuss the implementation and present experimental evaluation results. All our experiments were performed on a 32-bit system with an Intel Core 2 2.16 GHz CPU, 3.00 GB of memory, and the Windows 7 Professional operating system.

A. Cyber Resilience

Implementations. A unified XML [24] format was used to describe the power system control network topology and network access policy rules (e.g., firewall rules). During the offline phase, we use the NetAPT tool [25] to perform a comprehensive security analysis of the access policy rules and to produce the network connectivity matrix according to the control network topology input. The matrix is later translated to the corresponding CMDP model. On the power side, we used PowerWorld Simulator [26] to simulate the underlying system. In particular, we used the SimAuto toolbox to set up a real-time connection to PowerWorld.

In our experiments, we evaluated the solution on a simulated power grid infrastructure [4]. The control network models were built based on topology of a real power control network. Figure 2 shows the topology of a single control network that has 59 nodes, e.g., host systems and firewalls. The first control network monitors and controls buses 1 – 12 in the power system, and the second network monitors and controls buses 13 – 24. In particular, each power bus is monitored and controlled by a single host system in the corresponding control

2For presentation simplicity, the response and recovery transitions are not shown in Figure 1, but are discussed in Section VI-A.

network. Gray nodes show host systems; blue nodes are local network routers; red nodes represent firewalls; and the green node illustrates the organizational major router.

Model generation. Given the network topology and the access policy rules, i.e., about 100 firewall rules, the engine constructed the network connectivity matrix and generated the corresponding CMDP model. It is noteworthy that because the CMDP models may not be scalable specially for large-scale power-grid infrastructures, we use the envelope algorithm [27], where the CMDP is generated partially, and hence, not every individual state needs to be enumerated and analyzed. More technically, given the current system state, only reachable states up to some finite horizon are explored and used for the contingency analysis. Figure 3 illustrates a simplified version of the generated CMDP in which states with incidents that are exclusively cyber are drawn in white, while states with physical consequences are in gray. The first number on each state represents its ID. Table I maps each state ID to the IDs of the compromised assets in each state.

As shown on the generated CMDP, the attacker initially resides remotely in the Internet with no privilege on the power network (CMDP’s state 0) and can then traverse different attack paths to access a particular host in the power network. Each CMDP edge represents an access (i.e., possibly a vulnerability exploitation) allowed from a source to a destination host in the power network.

Metrics. In our experiments, we pessimistically assumed that all the hosts include security vulnerabilities in order to perform a worst case performance analysis. We calculated the resilience measure for individual states in the generated CMDP (i.e., shown as second and third number on each state in Figure 3, respectively). In a CMDP, there are usually many states with an identical set of physical contingencies that result in equal performance index values. To accelerate the metric calculation and minimize the number of connections to and calculations by PowerWorld, which is a time-consuming step
due to the AC power flow solution procedures, the implementation uses a caching solution to calculate the performance index value for each physical contingency set only once.

Incident ranking. We implemented the resilience metric to rank various security incidents that could occur according to the system’s current state and the generated CMDP model once the resilience indices are calculated for the power grid’s corresponding CMDP model. Figure 4 shows the time requirement of the index calculations for system models with different sizes. Table II shows the ranked list of cyber-physical contingencies for each state in our case study power grid. It is important to mention that the reported results are for the case in which the attacker has not yet caused any contingency in the power grid, i.e., the current state is $s_0 = \emptyset$ with ID 0. As shown, the edge $s_0\rightarrow s_4$ is ranked as the most critical contingency as it allows the attacker to get to the most impactful physical consequence with the least amount of cyber exploitation effort.

B. Physical Resilience

We evaluated our proposed resilience assessment metric through a numerical study using Matlab. Our study evaluated the resilience of the IEEE 57 bus power system test case [28] (Figure 5(a)) with linearized generator swing dynamics.

The states corresponded to the generator rotor angles. Since there are 7 generators in this system, the dimension of the state space is $N = 7$. In order to compute the state dynamics matrix $A$, we solved an optimal power flow problem to obtain an initial operating point using the Matpower utility [29]. This operating point is identical to the Matpower IEEE 57 bus test case, except with the load demands increased by 10% to create additional unstable modes (since stable linear systems are trivially resilient under our formulation). The generator dynamics were then computed by taking the Kron reduction of the admittance matrix to obtain an $N \times N$ admittance matrix $Y$, and then setting

$$\dot{x}_i(t) = \sum_{j \neq i} Y_{ij}E_jE_i\cos(\theta_i - \theta_j)(x_j - x_i),$$
TABLE II: Ranked (R) List of Security Incidents (CMDP edges) using the Resilience Metric

<table>
<thead>
<tr>
<th>R</th>
<th>Edge</th>
<th>R</th>
<th>Edge</th>
<th>R</th>
<th>Edge</th>
<th>R</th>
<th>Edge</th>
<th>R</th>
<th>Edge</th>
<th>R</th>
<th>Edge</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
<td>0.59</td>
<td>2</td>
<td>0.106</td>
<td>3</td>
<td>0.745</td>
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<td>10</td>
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<td>104</td>
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<td>105</td>
<td>0.026</td>
</tr>
</tbody>
</table>

**Note:** The table lists the ranked edges based on their resilience metrics. Each entry represents an edge in the IEEE 57 bus network, with the resilience metric indicating the importance of the edge for system resilience.

**Fig. 5:** Numerical evaluation of proposed resilience assessment framework. (a) IEEE 57 bus network used in the numerical study. The network contains 7 generators and 80 transmission lines. (b) Effect of variations in load at the given bus on the system resilience to compromise of any single generator. The cost of recovery fluctuates as the load increases. (c) Effect of removing transmission lines on the system resilience. The impact on resilience varies between transmission lines, and in some cases causes the unstable modes to become uncontrollable, thus violating the definition of resilience.

**TABLE III:** Evaluation of resilience metric for each possible set of compromised controllers.

<table>
<thead>
<tr>
<th>Set S</th>
<th>Cost</th>
<th>Set S</th>
<th>Cost</th>
<th>Set S</th>
<th>Cost</th>
<th>Set S</th>
<th>Cost</th>
<th>Set S</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
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<td>1.42 × 10^8</td>
<td>{1}</td>
<td>1.23 × 10^8</td>
<td>{1}</td>
<td>2.72 × 10^7</td>
<td>{2}</td>
<td>2.42 × 10^7</td>
</tr>
<tr>
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<td>2.72 × 10^7</td>
<td>{2}</td>
<td>2.42 × 10^7</td>
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<td>1.40 × 10^8</td>
</tr>
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<td>2.42 × 10^7</td>
<td>{1}</td>
<td>1.40 × 10^8</td>
<td>{1}</td>
<td>1.40 × 10^8</td>
</tr>
<tr>
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<td>{2}</td>
<td>2.72 × 10^7</td>
<td>{6}</td>
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**Cost of recovery after edge deletion**

**Cost of recovery after increase in load**
where $E_i$ and $E_j$ are the excitations of generators $i$ and $j$, and $\theta_i$ and $\theta_j$ are the rotor angles at the power flow solution. The control matrix $B$ was modeled as the $N \times N$ identity matrix.

We first computed the recovery cost for this system, for each set of compromised generators $S$, using the matrices $Q = R = I$. The results are summarized in Table III. We observe that the cost of recovery is monotonically increasing in the set of compromised controllers, i.e., if $S' \subseteq S'$ then the cost of recovery from compromise of $S'$ exceeds the cost of recovery from $S$.

The resilience of the system also depends on the power system operating conditions. We analyzed two of those factors, namely, the amount of system load and the system topology. The impact of changing the system load on the worst-case cost of recovery from compromising any single generator is shown in Figure 5(b). In this figure, the load at the bus with given index is doubled and the change in resilience is observed. Increasing the load had little impact on the resilience. On the other hand, deleting edges from the network topology resulted in a large increase in the recovery cost, with significant variation between edges, in some cases causing the system to become non-resilient.

**IX. CONCLUSIONS**

We presented a formal definition of cyber-physical resilience, and a metric to quantify the resilience level of a given cyber-physical system. Our metric leverages the conceptual ideas from the resilience terminology used in material science, and considers the resilience in both cyber and physical components as well as their interdependencies. Our proposed metric uses discrete stochastic models (competitive Markov decision processes) to encode the dynamics and interdependencies of the cyber network along with the dynamical linear system models to capture the continuous dynamics of the underlying physical processes. We implemented a prototype of the proposed metric for case study systems, and demonstrate its capabilities for joint consideration of the cyber and physical incidents.

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